

## NEW QCD-ESTIMATE OF THE KAON PENGUIN MATRIX ELEMENTS AND $\epsilon'/\epsilon$

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Firstly, we use the recent ALEPH/OPAL data on the  $V - A$  spectral functions for fixing the continuum threshold with which the first and second Weinberg sum rules should be satisfied in the chiral limit. Then, we predict the values of the low-energy constants  $m_{\pi^+} - m_{\pi^0}$ ,  $L_{10}$ , and test the values of the electroweak kaon penguin matrix elements  $\langle Q_{8,7}^{3/2} \rangle_{2\pi}$  obtained from DMO-like sum rules. Secondly, we use the data on the  $\tau$ -total hadronic width  $R_{\tau,V/A}$  for extracting  $\langle Q_8^{3/2} \rangle_{2\pi}$ , in the  $\overline{MS}$ -scheme, and propose some new sum rules for  $\langle Q_7^{3/2} \rangle_{2\pi}$  in the chiral limit, where the latter require more accurate data for the spectral functions near the  $\tau$ -mass. Thirdly, we analyze the effects to the matrix element  $\langle Q_6^{1/2} \rangle_{2\pi}$ , of the  $S_2 \equiv (\bar{u}u + \bar{d}d)$  component of the  $I = 0$  scalar meson, with its parameters fixed from QCD spectral sum rules. Our results should stimulate a further attention on the rôle of the (expected large) gluonium component of the  $I = 0$  scalar meson and of the associated operator in the  $K \rightarrow \pi\pi$  amplitude. Finally, using our previous determinations, we deduce, in the Standard Model (SM), the conservative upper bound for the CP-violating ratio:  $\epsilon'/\epsilon \leq (22 \pm 9)10^{-4}$ , which is in agreement with the present measurements.

## 1. INTRODUCTION AND GENERALITIES

$CP$ -violation is one of the most important weak interactions phenomena in particle physics [1,2], where in their presence, stable (under strong interactions)  $K^0(\bar{s}d)$  and  $\bar{K}^0(\bar{d}s)$  particles with definite strangeness eigenvalues  $\pm 1$  become unstable. Therefore, the decay of a long-lived kaon <sup>1</sup> into a two-pion final state is an evidence for  $CP$ -violation. The first observation of such a transition to the  $\pi^+\pi^-$  mode, was discovered 36 years ago by [3] from  $K^0$ - $\bar{K}^0$  mixing. Since then, the transition into  $\pi^0\pi^0$  and the phases of the ratio of the amplitudes:

$$\eta_{+-} \equiv \frac{A(K_L \rightarrow \pi^+\pi^-)}{A(K_S \rightarrow \pi^+\pi^-)}, \quad \eta_{00} \equiv \frac{A(K_L \rightarrow \pi^0\pi^0)}{A(K_S \rightarrow \pi^0\pi^0)}, \quad (1)$$

have been also observed. For a phenomenological analysis, it is convenient to work with quantities where the final pion states are in a definite isospin. Then, one introduces the indirect (through  $K^0$ - $\bar{K}^0$  mixing)  $CP$ -violation parameter  $\epsilon$ , and the quantity  $\omega$  governing the so-called  $\Delta = 1/2$  rule (enhancement of the  $I = 0$  over the  $I = 2$  transitions):

$$\epsilon \equiv \frac{A[K_L \rightarrow (\pi\pi)_{I=0}]}{A[K_S \rightarrow (\pi\pi)_{I=0}]}, \quad \omega \equiv \frac{A[K_S \rightarrow (\pi\pi)_{I=2}]}{A[K_S \rightarrow (\pi\pi)_{I=0}]}, \quad (2)$$

and the direct (in the amplitude)  $CP$ -violation parameter:

$$\epsilon' \equiv \frac{1}{\sqrt{2}} \left\{ \frac{A[K_L \rightarrow (\pi\pi)_{I=2}]}{A[K_S \rightarrow (\pi\pi)_{I=0}]} - \epsilon \times \omega \right\}. \quad (3)$$

In terms of these quantities, one can express the measured  $\eta_{+-}$  and  $\eta_{00}$  quantities as:

$$\eta_{+-} = \epsilon + \frac{\epsilon'}{1 + \omega/\sqrt{2}}, \quad \eta_{00} = \epsilon - \frac{2\epsilon'}{1 - \sqrt{2}\omega}, \quad (4)$$

from which one can deduce the experimental value [4]:

$$\epsilon \simeq (2.280 \pm 0.013) \times 10^{-3} e^{i(43.5 \pm 0.1)^\circ}. \quad (5)$$

To proceed further, one introduces the isospin amplitude:

$$A(K^0 \rightarrow (\pi\pi)_I) = iA_I e^{i\delta_I}, \quad A(\bar{K}^0 \rightarrow (\pi\pi)_I) = -iA_I^* e^{i\delta_I}, \quad (6)$$

and makes use of the mass-difference  $\Delta m \equiv m_L - m_S$ , between  $K_S$  and  $K_L$ , and of  $M_{12}$  (off-diagonal dispersive part of the  $K^0$ - $\bar{K}^0$  complex mass matrix:  $\text{Re } M_{12} \simeq \Delta m/2$ ). Therefore, one can write, within a good approximation [1]:

$$\epsilon \simeq \frac{1}{\sqrt{2}} e^{i\frac{\pi}{4}} \left( \frac{\text{Im } M_{12}}{\Delta m} + \frac{\text{Im } A_0}{\text{Re } A_0} \right), \quad \omega \approx e^{i(\delta_2 - \delta_0)} \frac{\text{Re } A_2}{\text{Re } A_0}. \quad (7)$$

and:

$$\epsilon' \simeq \frac{1}{\sqrt{2}} e^{i(\delta_2 - \delta_0 + \frac{\pi}{2})} \frac{\text{Re } A_2}{\text{Re } A_0} \left( \frac{\text{Im } A_2}{\text{Re } A_2} - \frac{\text{Im } A_0}{\text{Re } A_0} \right) \quad (8)$$

Experimentally [4],  $\delta_2 - \delta_0 \simeq -(42 \pm 4)^\circ$ , and then, using the data on  $\Gamma(K_S \rightarrow \pi^+\pi^-)/\Gamma(K_S \rightarrow \pi^0\pi^0)$ , one can deduce:

$$\omega_{exp} \simeq \left( \frac{1}{22} \right) e^{-i(42 \pm 4)^\circ}, \quad (9)$$

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<sup>1</sup> $|K_L\rangle$  and  $|K_S\rangle$  are very close to the  $CP$ -eigenstates  $|K_1^0\rangle \equiv \frac{1}{\sqrt{2}}(|K^0\rangle - |\bar{K}^0\rangle)$  and  $|K_2^0\rangle \equiv \frac{1}{\sqrt{2}}(|K^0\rangle + |\bar{K}^0\rangle)$  with  $CP|K_1^0\rangle = +|K_1^0\rangle$  and  $CP|K_2^0\rangle = -|K_2^0\rangle$ . Their lifetime are  $\tau_L \simeq 5.2 \times 10^{-8}$  sec  $\approx 15.51$  m  $\approx 5.8 \times 10^2 \tau_S$ . Their mass-difference is  $\Delta m \equiv M_L - M_S = (3.522 \pm 0.016) \times 10^{-12}$  MeV.

while the recent measurement reported by the kTeV and NA48 experiments [5] on the direct  $CP$ -violation ratio is <sup>2</sup>:

$$\text{Re}\left(\frac{\epsilon'}{\epsilon}\right)_{exp} \simeq (21.4 \pm 4.0) \times 10^{-4} . \quad (10)$$

It is fair to say that a simultaneous explanation of these two previous experimental numbers remains a challenge for the present theoretical predictions within the Standard Model (SM) [1,2,7-10].

## 2. THEORY OF $\epsilon'/\epsilon$

In the SM, it is customary to study the  $\Delta S = 1$  process from the weak hamiltonian:

$$\mathcal{H}_{eff} = \frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \sum_{i=1}^{10} C_i(\mu) Q_i(\mu) , \quad (11)$$

where  $C_i(\mu)$  are perturbative Wilson Coefficients known including complete NLO QCD corrections [7], which read in the notations of [7]:

$$C_i(\mu) \equiv z_i(\mu) - \frac{V_{td} V_{ts}^*}{V_{ud} V_{us}^*} y_i(\mu) . \quad (12)$$

where  $V_{ij}$  are elements of the CKM-matrix;  $Q_i(\mu)$  are non-perturbative hadronic matrix elements which need to be estimated from different non-perturbative methods of QCD (chiral perturbation theory, lattice, QCD spectral sum rules,...). In the choice of basis of [7], the dominant contributions come from the four-quark operators which are classified as:

- **Current-Current:**

$$\mathcal{Q}_1 \equiv (\bar{s}_\alpha u_\beta)_{V-A} (\bar{u}_\beta d_\alpha)_{V-A} , \quad \mathcal{Q}_2 \equiv (\bar{s}u)_{V-A} (\bar{u}d)_{V-A} \quad (13)$$

- **QCD-penguins:**

$$\begin{aligned} \mathcal{Q}_3 &\equiv (\bar{s}d)_{V-A} \sum_{u,d,s} (\bar{\psi}\psi)_{V-A} , & \mathcal{Q}_4 &\equiv (\bar{s}_\alpha d_\beta)_{V-A} \sum_{u,d,s} (\bar{\psi}_\beta \psi_\alpha)_{V-A} , \\ \mathcal{Q}_5 &\equiv (\bar{s}d)_{V-A} \sum_{u,d,s} (\bar{\psi}\psi)_{V+A} , & \mathcal{Q}_6 &\equiv (\bar{s}_\alpha d_\beta)_{V-A} \sum_{u,d,s} (\bar{\psi}_\beta \psi_\alpha)_{V+A} \end{aligned} \quad (14)$$

- **Electroweak-penguins:**

$$\begin{aligned} \mathcal{Q}_7 &\equiv \frac{3}{2} (\bar{s}d)_{V-A} \sum_{u,d,s} e_\psi (\bar{\psi}\psi)_{V+A} , & \mathcal{Q}_8 &\equiv \frac{3}{2} (\bar{s}_\alpha d_\beta)_{V-A} \sum_{u,d,s} e_\psi (\bar{\psi}_\beta \psi_\alpha)_{V+A} , \\ \mathcal{Q}_9 &\equiv \frac{3}{2} (\bar{s}d)_{V-A} \sum_{u,d,s} e_\psi (\bar{\psi}\psi)_{V-A} , & \mathcal{Q}_{10} &\equiv \frac{3}{2} (\bar{s}_\alpha d_\beta)_{V-A} \sum_{u,d,s} e_\psi (\bar{\psi}_\beta \psi_\alpha)_{V-A} , \end{aligned} \quad (15)$$

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<sup>2</sup>If one includes the preliminary NA48 result  $\text{Re}(\epsilon'/\epsilon) \simeq (12.2 \pm 2.9(stat.) \pm 4(syst.)) \times 10^{-4}$  from the 98 data sample, the preliminary new experimental world average becomes  $\text{Re}(\epsilon'/\epsilon) \simeq (19.3 \pm 2.4) \times 10^{-4}$  [6].

where  $\alpha, \beta$  are colour indices;  $e_\psi$  denotes the electric charges<sup>3</sup> reflecting the electroweak nature of  $\mathcal{Q}_{7,\dots,10}$ , while  $V - (+)A \equiv (1 - (+)\gamma_5)\gamma_\mu$ . Using an OPE of the amplitudes, one obtains:

$$\frac{\epsilon'}{\epsilon} \simeq \text{Im}\lambda_t \left[ P^{(1/2)} - P^{(3/2)} \right] e^{i\Phi} \quad (16)$$

where  $\Phi \equiv \Phi_{\epsilon'} - \Phi_\epsilon \approx 0$  [see Eqs. (7) and (8)];  $\lambda_t \equiv V_{td}V_{ts}^*$  can be expressed in terms of the CKM matrix elements as ( $\delta$  being the CKM phase) [7,11]:

$$\text{Im}\lambda_t \approx |V_{ub}||V_{cb}|\sin\delta \simeq (1.33 \pm 0.14) \times 10^{-4}, \quad (17)$$

from  $B$ -decays and  $\epsilon$ . The QCD quantities  $P^{(I)}$  read:

$$\begin{aligned} P^{(1/2)} &= \frac{G_F |\omega|}{2|\epsilon|\text{Re}A_0} \sum_i C_i(\mu) \langle (\pi\pi)_{I=0} | \mathcal{Q}_i | K^0 \rangle_0 (1 - \Omega_{IB}), \\ P^{(3/2)} &= \frac{G_F}{2|\epsilon|\text{Re}A_2} \sum_i C_i(\mu) \langle (\pi\pi)_{I=2} | \mathcal{Q}_i | K^0 \rangle_2. \end{aligned} \quad (18)$$

$\Omega_{IB} \simeq (0.16 \pm 0.03)$  quantifies the  $SU(2)$ -isospin breaking effect, which includes the one of the  $\pi^0$ - $\eta$  mixing [12], and which reduces the usual value of  $(0.25 \pm 0.08)$  [7] due to  $\eta'$ - $\eta$  mixing. It is also expected that the QCD- and electroweak-penguin operators:

$$\mathcal{Q}_8^{3/2} \approx B_8^{3/2}/m_s^2 + \mathcal{O}(1/N_c), \quad \mathcal{Q}_6^{1/2} \approx B_6^{1/2}/m_s^2 + \mathcal{O}(1/N_c), \quad (19)$$

give the dominant contributions to the ratio  $\epsilon'/\epsilon$  [13];  $B$  are the bag factors which are expected to be 1 in the large  $N_c$ -limit. Therefore, a simplified approximate but very informative expression of the theoretical predictions can be derived [7]:

$$\frac{\epsilon'}{\epsilon} \approx 13 \text{Im}\lambda_t \left( \frac{110}{\overline{m}_s(2) [\text{MeV}]} \right)^2 \times \left[ B_6^{1/2} (1 - \Omega_{IB}) - 0.4 B_8^{3/2} \left( \frac{m_t}{165 \text{ GeV}} \right) \right] \left( \frac{\Lambda_{\overline{MS}}^{(4)}}{340 \text{ MeV}} \right), \quad (20)$$

where the average value  $\hat{B}_K = 0.80 \pm 0.15$  of the  $\Delta S = 2$  process has been used. This value includes the conservative value  $0.58 \pm 0.22$  from Laplace sum rules [14]. The values of the top quark mass and the QCD scale  $\Lambda_{\overline{MS}}^{(4)}$  [4,15] are under a quite good control and have small effects. A recent review of the light quark mass determinations [16] also indicates that the strange quark mass is also under control and a low value advocated in the previous literature to explain the present data on  $\epsilon'/\epsilon$  is unlikely due the lower bound constraints from the positivity of the QCD spectral function or from the positivity of the  $m^2$  corrections to the GMOR PCAC relation. For a consistency with the approach used in this paper, we shall use the average value of the light quark masses from QCD spectral sum rules(QSSR),  $e^+e^-$  and  $\tau$ -decays given in [16]:

$$\overline{m}_s(2) \simeq (119 \pm 12) \text{ MeV}, \quad \overline{m}_d(2) \simeq (6.3 \pm 0.8) \text{ MeV}, \quad \overline{m}_u(2) \simeq (3.5 \pm 0.4) \text{ MeV}. \quad (21)$$

Using the previous experimental values, one can deduce the constraint in [16] updated:

$$\mathcal{B}_{68} \equiv B_6^{1/2} - 0.48 B_8^{3/2} \simeq 1.73 \pm 0.50 \text{ (resp. } \geq 1.0 \sim 1.2), \quad (22)$$

if one uses the value of  $m_s$  in Eq. (21) (resp. the lower bound of  $(90 \sim 100)$  MeV reported in [16]). This result shows a possible violation of more than  $2\sigma$  for the leading  $1/N_c$  vacuum saturation prediction  $\approx 0.52$  corresponding to  $B_6^{1/2} \approx B_8^{3/2} \approx 1$ . Consulting the available predictions reviewed in [7], which we will summarize and update in Table 1, one can notice that

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<sup>3</sup>Though apparently suppressed, the effect of the electroweak penguins are enhanced by  $1/\omega$  as we shall see later on in Eq. (18).

Table 1

Penguin  $B$ -parameters for the  $\Delta S = 1$  process from different approaches at  $\mu = 2$  GeV. We use the value  $m_s(2) = (119 \pm 12)$  MeV from [16], and predictions based on dispersion relations [25,24] have been rescaled according to it. We also use for our results  $f_\pi = 92.4$  MeV [4], but we give in the text their  $m_s$  and  $f_\pi$  dependences. Results without any comments on the scheme have been obtained in the  $\overline{MS} - NDR$ -scheme. However, at the present accuracy, one cannot differentiate these results from the ones of  $\overline{MS} - HV$ -scheme.

METHODS	$B_6^{1/2}$	$B_8^{3/2}$	$B_7^{3/2}$	COMMENTS
Lattice [9,17,18]	$0.6 \sim 0.8$ unreliable	$0.7 \sim 1.1$	$0.5 \sim 0.8$	Huge NLO at matching[19]
Large $N_c$ [20]	$0.7 \sim 1.3$  $1.5 \sim 1.7$	$0.4 \sim 0.7$  —	$-0.10 \sim 0.04$  —	$\mathcal{O}(p^0/N_c, p^2)$ scheme? $\mathcal{O}(p^2/N_c)$ ; $m_q = 0$ scheme?
<b>Models</b>				
Chiral QM [10]	$1.2 \sim 1.7$	$\sim 0.9$	$\approx B_8^{3/2}$	$\mu = .8$ GeV rel. with $\overline{MS}$ ?
ENJL+IVB [21]	$2.5 \pm 0.4$	$1.4 \pm 0.2$	$0.8 \pm 0.1$	$NLO$ in $1/N_c$ $m_q = 0$
$L\sigma$ -model [22]	$\sim 2$	$\sim 1.2$	—	Not unique $\mu \approx 1$ GeV; scheme ?
NL $\sigma$ -model [23]	$1.6 \sim 3.0$	$0.7 \sim 0.9$	—	$M_\sigma$ : free; $SU(3)_F$ trunc. $\mu \approx 1$ GeV; scheme ?
<b>Dispersive</b>				
Large $N_c$ + LMD +LSD-match.[24]	—	—	$0.9$ strong $\mu$ -dep.	$NLO$ in $1/N_c$ ,
DMO-like SR [25]	—	$1.6 \pm 0.4$ huge NLO	$0.8 \pm 0.2$	$m_q = 0$ Strong $s$ , $\mu$ -dep.
FSI [26]	$1.4 \pm 0.3$	$0.7 \pm 0.2$	—	Debate for fixing the Slope [27]
<b>This work</b>				
DMO-like SR: [25] revisited	—	$2.2 \pm 1.5$ inaccurate	$0.7 \pm 0.2$	$m_q = 0$ Strong $s$ , $\mu$ -dep.
$\tau$ -like SR	—	—	inaccurate	$t_c$ -changes
$R_\tau^{V-A}$	—	$1.7 \pm 0.4$	—	$m_q = 0$
$S_2 \equiv (\bar{u}u + \bar{d}d)$ from QSSR	$1.0 \pm 0.4$ $\leq 1.5 \pm 0.4$	—	—	$\overline{MS}$ -scheme $m_s(2) \geq 90$ MeV

the values of the  $B$ -parameters have large errors. One can also see that results from QCD first principles (lattice and  $1/N_c$ ) fail to explain the data, which however can be accomodated by various QCD-like models. We shall come back to this discussion when we shall compare our results with presently available predictions. It is, therefore, clear that the present estimate of the four-quark operators, and in particular the estimates of the dominant penguin ones given previously in Eq. (19), need to be reinvestigated. Due to the complex structures and large size of these operators, they should be difficult to extract unambiguously from different approaches. In this paper, we present alternative theoretical approaches based as well on first principles of QCD ( $\tau$ -decay data, analyticity), for predicting the size of the QCD- and electroweak-penguin operators given in Eq. (19). In performing this analysis, we shall also encounter the electroweak penguin operator:

$$Q_7^{3/2} \approx B_7^{3/2}/m_s^2 + \mathcal{O}(1/N_c) , \quad (23)$$

and some other low-energy constants ( $m_{\pi^\pm} - m_{\pi^0}$ ,  $L_{10}$ ) though not directly relevant to  $\epsilon'/\epsilon$ .

### 3. TESTS OF THE “SACROSANTE” WEINBERG AND DMO SUM RULES IN THE CHIRAL LIMIT

#### 3.1. Notations

Before estimating these condensates, we shall test the procedure used in [25] by analyzing the classics DMO- and Weinberg-like sum rules [28,29]. This analysis will also allow us to fix the cut-off parameter  $t_c$  until which the data on  $V - A$  spectral functions from ALEPH/OPAL [30,31] are known. We shall be concerned here with the two-point correlator:

$$\Pi_{LR}^{\mu\nu}(q) \equiv i \int d^4x e^{iqx} \langle 0 | T J_L^\mu(x) (J_R^\nu(0))^\dagger | 0 \rangle = -(g^{\mu\nu} q^2 - q^\mu q^\nu) \Pi_{LR}(q^2) , \quad (24)$$

built from the left- and right-handed components of the local weak current:

$$J_L^\mu = \bar{u} \gamma^\mu (1 - \gamma_5) d, \quad J_R^\mu = \bar{u} \gamma^\mu (1 + \gamma_5) d , \quad (25)$$

and/or using isospin rotation relating the neutral and charged weak currents:

$$\rho_V - \rho_A \equiv \frac{1}{2\pi} \text{Im} \Pi_{LR} \equiv \frac{1}{4\pi^2} (v - a) . \quad (26)$$

The first term is the notation in [25], while the last one is the notation in [30,31].

#### 3.2. The sum rules

The “sacrosante” DMO and Weinberg sum rules read in the chiral limit <sup>4</sup>:

$$\begin{aligned} \mathcal{S}_0 &\equiv \int_0^\infty ds \frac{1}{2\pi} \text{Im} \Pi_{LR} = f_\pi^2 , \\ \mathcal{S}_1 &\equiv \int_0^\infty ds s \frac{1}{2\pi} \text{Im} \Pi_{LR} = 0 , \\ \mathcal{S}_{-1} &\equiv \int_0^\infty \frac{ds}{s} \frac{1}{2\pi} \text{Im} \Pi_{LR} = -4L_{10} , \\ \mathcal{S}_{em} &\equiv \int_0^\infty ds \left( s \log \frac{s}{\mu^2} \right) \frac{1}{2\pi} \text{Im} \Pi_{LR} = -\frac{4\pi}{3\alpha} f_\pi^2 (m_{\pi^\pm}^2 - m_{\pi^0}^2) , \end{aligned} \quad (27)$$

where  $f_\pi|_{exp} = (92.4 \pm 0.26)$  MeV is the experimental pion decay constant which should be used here as we shall use data from  $\tau$ -decays involving physical pions;  $m_{\pi^\pm} - m_{\pi^0}|_{exp} \simeq 4.5936(5)$  MeV;

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<sup>4</sup>Systematic analysis of the breaking of these sum rules by light quark masses [32] and condensates [33,34] within the context of QCD have been done earlier.

$L_{10} \equiv f_\pi^2 \langle r_\pi^2 \rangle / 3 - F_A [\langle r_\pi^2 \rangle] = (0.439 \pm 0.008) fm^2$  is the mean pion radius and  $F_A = 0.0058 \pm 0.0008$  is the axial-vector pion form factor for  $\pi \rightarrow e\nu\gamma$  is one the low-energy constants of the effective chiral Lagrangian [1]. In order to exploit these sum rules using the ALEPH/OPAL [30,31] data from the hadronic tau-decays, we shall work with their Finite Energy Sum Rule (FESR) versions (see e.g. [32,35] for such a derivation). In the chiral limit ( $m_q = 0$  and  $\langle \bar{u}u \rangle = \langle \bar{d}d \rangle = \langle \bar{s}s \rangle$ ), this is equivalent to truncate the LHS at  $t_c$  until which the data are available, while the RHS of the integral remains valid to leading order in the  $1/t_c$  expansion in the chiral limit, as the breaking of these sum rules by higher dimension  $D = 6$  condensates in the chiral limit which is of the order of  $1/t_c^3$  is numerically negligible [33].

### 3.3. Matching between the low and high-energy regions

In order to fix the  $t_c$  values which separate the low and high energy parts of the spectral functions, we require that the 2nd Weinberg sum rule (WSR)  $\mathcal{S}_1$  should be satisfied by the present data.

Figure 1. FESR version of the 2nd Weinberg sum rule versus  $t_c$  in  $\text{GeV}^2$  using the ALEPH/OPAL data of the spectral functions. Only the central values are shown.

As shown in Fig. 1, this is obtained for two values of  $t_c$  <sup>5</sup>:

$$t_c \simeq (1.4 \sim 1.5) \text{ GeV}^2 \quad \text{and} \quad t_c \simeq (2.4 \sim 2.6) \text{ GeV}^2. \quad (28)$$

Though the 2nd value is interesting from the point of view of the QCD perturbative calculations (better convergence of the QCD series), its exact value is strongly affected by the inaccuracy of the data near the  $\tau$ -mass (with the low values of the ALEPH/OPAL data points, the 2nd Weinberg sum rule is only satisfied at the former value of  $t_c$ ).

After having these  $t_c$  solutions, we can improve the constraints by requiring that the 1st Weinberg sum rule  $\mathcal{S}_0$  reproduces the experimental value of  $f_\pi$  <sup>6</sup> within an accuracy 2-times the experimental error. This condition allows to fix  $t_c$  in a very narrow margin due to the sensitivity of the result on the changes of  $t_c$  values <sup>7</sup>

$$t_c = (1.475 \pm 0.015) \text{ GeV}^2, \quad (29)$$

## 4. LOW-ENERGY CONSTANTS $L_{10}$ , $m_{\pi^\pm} - m_{\pi^0}$ AND $f_\pi$ IN THE CHIRAL LIMIT

Using the previous value of  $t_c$  into the  $\mathcal{S}_{-1}$  sum rule, we deduce:

$$L_{10} \simeq -(6.26 \pm 0.04) \times 10^{-3}, \quad (30)$$

which agrees quite well with more involved analysis including chiral symmetry breakings [37,31], and with the one using a lowest meson dominance (LMD) of the spectral integral [24].

Analogously, one obtains from the  $\mathcal{S}_{em}$  sum rule:

$$\Delta m_\pi \equiv m_{\pi^\pm} - m_{\pi^0} \simeq (4.84 \pm 0.21) \text{ MeV}. \quad (31)$$

<sup>5</sup>One can compare the two solutions with the  $t_c$ -stability region around 2  $\text{GeV}^2$  in the QCD spectral sum rules analysis (see e.g. Chapter 6 of [36]).

<sup>6</sup>Though we are working here in the chiral limit, the data are obtained for physical pions, such that the corresponding value of  $f_\pi$  should also correspond to the experimental one.

<sup>7</sup>For the second set of  $t_c$ -values in Eq. 28, one obtains a slightly lower value:  $f_\pi = (84.1 \pm 4.4) \text{ MeV}$ .

This result is  $1\sigma$  higher than the data  $4.5936(5)$  MeV, but agrees within the errors with the more detailed analysis from  $\tau$ -decays [38,31] and with the LMD result of about 5 MeV [24]. We have checked that moving the subtraction point  $\mu$  from 2 to 4 GeV slightly decreases the value of  $\Delta m_\pi$  by 3.7% which is relatively weak, as expected. Indeed, in the chiral limit, the  $\mu$  dependence does not appear (to leading order in  $a_s$ ) in the RHS of the  $\mathcal{S}_{em}$  sum rule, and then, it looks natural to choose:

$$\mu^2 = t_c , \quad (32)$$

as  $t_c$  is the only external scale in the analysis. At this scale the result increases slightly by 2.5%. One can also notice that the prediction for  $\Delta m$  is more stable when one changes the value of  $t_c = \mu^2$ . Therefore, the final predictions from the value of  $t_c$  in Eq. (29) fixed from the 1st and 2nd Weinberg sum rules are:

$$\Delta m \simeq (4.96 \pm 0.22) \text{ MeV} , \quad L_{10} \simeq -(6.42 \pm 0.04) \times 10^{-3} , \quad (33)$$

which we consider as our "best" predictions.

For some more conservative results, we also give the predictions obtained from the second  $t_c$ -value given in Eq. (28). In this way, one obtains:

$$f_\pi = (87 \pm 4) \text{ MeV} , \quad \Delta m \simeq (3.4 \pm 0.3) \text{ MeV} , \quad L_{10} \simeq -(5.91 \pm 0.08) \times 10^{-3} , \quad (34)$$

where one can notice that the results are systematically lower than the ones obtained in Eq. (33) from the first  $t_c$ -value given previously, which may disfavour a posteriori the second choice of  $t_c$ -values, though we do not have a strong argument favouring one with respect to the other <sup>8</sup>. Therefore, we take as a conservative value the largest range spanned by the two sets of results, namely:

$$f_\pi = (86.8 \pm 7.1) \text{ MeV} , \quad \Delta m \simeq (4.1 \pm 0.9) \text{ MeV} , \quad L_{10} \simeq -(5.8 \pm 0.2) \times 10^{-3} , \quad (35)$$

which we found to be quite satisfactory in the chiral limit. The previous tests are very useful, as they will allow us to gauge the confidence level of the next predictions.

## 5. SOFT PION AND KAON REDUCTIONS OF $\langle(\pi\pi)_{I=2}|\mathcal{Q}_{7,8}^{3/2}|K^0\rangle$

An interesting approach combining pion and kaon reductions in the chiral limit with dispersion relation techniques have been proposed recently [25] <sup>9</sup> in order to estimate the matrix element:

$$\langle\mathcal{Q}_{7,8}^{3/2}\rangle_{2\pi} \equiv \langle(\pi\pi)_{I=2}|\mathcal{Q}_{7,8}^{3/2}|K^0\rangle . \quad (36)$$

In the chiral limit  $m_{u,d,s} \sim m_\pi^2 \simeq m_K^2 = 0$ , one can use soft pion and kaon techniques in order to relate the previous amplitude to the four-quark vacuum condensates:

$$\begin{aligned} \langle\mathcal{Q}_7^{3/2}\rangle_{2\pi} &\simeq -\frac{4}{f_\pi^3}\langle\mathcal{O}_7^{3/2}\rangle , \\ \langle\mathcal{Q}_8^{3/2}\rangle_{2\pi} &\simeq -\frac{4}{f_\pi^3}\left\{\frac{1}{3}\langle\mathcal{O}_7^{3/2}\rangle + \frac{1}{2}\langle\mathcal{O}_8^{3/2}\rangle\right\} , \end{aligned} \quad (37)$$

<sup>8</sup>Approach based on  $1/N_c$  expansion and a saturation of the spectral function by the lowest state within a narrow width approximation (NWA) favours the former value of  $t_c$  given in Eq. (29) [?].

<sup>9</sup>A similar approach based on large  $N_c$  and a lowest meson dominance (LMD) of the spectral functions is also done in [24].



where we use the shorthand notations:  $\langle 0|\mathcal{O}_{7,8}^{3/2}|0\rangle \equiv \langle \mathcal{O}_{7,8}^{3/2}\rangle$ , and  $f_\pi = (92.42 \pm 0.26) \text{ MeV}$ <sup>10</sup>. Here:

$$\begin{aligned}\mathcal{O}_7^{3/2} \equiv \mathcal{O}_1^{3/2}[25] &= \sum_{u,d,s} \bar{\psi}\gamma_\mu \frac{\tau_3}{2} \psi \bar{\psi}\gamma_\mu \frac{\tau_3}{2} \psi - \bar{\psi}\gamma_\mu \gamma_5 \frac{\tau_3}{2} \psi \bar{\psi}\gamma_\mu \gamma_5 \frac{\tau_3}{2} \psi , \\ \mathcal{O}_8^{3/2} &= \sum_{u,d,s} \bar{\psi}\gamma_\mu \lambda_a \frac{\tau_3}{2} \psi \bar{\psi}\gamma_\mu \lambda_a \frac{\tau_3}{2} \psi - \bar{\psi}\gamma_\mu \gamma_5 \lambda_a \frac{\tau_3}{2} \psi \bar{\psi}\gamma_\mu \gamma_5 \lambda_a \frac{\tau_3}{2} \psi ,\end{aligned}\quad (38)$$

where  $\tau_3$  and  $\lambda_a$  are flavour and colour matrices. Using further pion and kaon reductions in the chiral limit, one can relate this matrix element to the  $B$ -parameters:

$$\begin{aligned}B_7^{3/2}(M_\tau^2) &\simeq \frac{3}{4} \frac{(m_u + m_d)}{m_\pi^2} \frac{(m_u + m_s)}{m_K^2} \frac{1}{f_\pi} \langle \mathcal{Q}_7^{3/2} \rangle_{2\pi}(M_\tau^2) \\ B_8^{3/2}(M_\tau^2) &\simeq \frac{1}{4} \frac{(m_u + m_d)}{m_\pi^2} \frac{(m_u + m_s)}{m_K^2} \frac{1}{f_\pi} \langle \mathcal{Q}_8^{3/2} \rangle_{2\pi}(M_\tau^2)\end{aligned}\quad (39)$$

where all QCD quantities will be evaluated in the  $\overline{MS}$ -scheme and at the scale  $M_\tau$ .

## 6. $\langle (\pi\pi)_{I=2} | \mathcal{Q}_{7,8}^{3/2} | K^0 \rangle$ FROM DMO-LIKE SUM RULES IN THE CHIRAL LIMIT

In a previous paper [25], the vacuum condensates  $\langle \mathcal{O}_{7,8}^{3/2} \rangle$  which are related to the weak matrix elements  $\langle (\pi\pi)_{I=2} | \mathcal{Q}_{7,8}^{3/2} | K^0 \rangle$  through the soft pion and kaon reduction techniques (see previous section) have been extracted using Das-Mathur-Okubo(DMO)- and Weinberg-like sum rules based on the difference of the vector and axial-vector spectral functions  $\rho_{V,A}$  of the  $I = 1$  component of the neutral current:

$$\begin{aligned}2\pi \langle \alpha_s \mathcal{O}_8^{3/2} \rangle(\mu^2) &= \int_0^\infty ds \, s^2 \frac{\mu^2}{s + \mu^2} (\rho_V - \rho_A)(s) , \\ \frac{16\pi^2}{3} \langle \mathcal{O}_7^{3/2} \rangle(\mu^2) &= \int_0^\infty ds \, s^2 \log \left( \frac{s + \mu^2}{s} \right) (\rho_V - \rho_A)(s) ,\end{aligned}\quad (40)$$

where  $\mu$  is the subtraction point. Due to the quadratic divergence of the integrand, the previous sum rules are expected to be sensitive to the high energy tails of the spectral functions where the present ALEPH/OPAL data from  $\tau$ -decay [30,31] are inaccurate. This inaccuracy can a priori affect the estimate of the four-quark vacuum condensates. On the other hand, the explicit  $\mu$ -dependence of the analysis can also induce another uncertainty. En passant, we check below the effects of these two parameters  $t_c$  and  $\mu$ . After evaluating the spectral integrals, we obtain at  $\mu = 2 \text{ GeV}$  and for our previous values of  $t_c$  in Eq. (29), the values (in units of  $10^{-3} \text{ GeV}^6$ ) using the cut-off momentum scheme (c.o):

$$\alpha_s \langle \mathcal{O}_8^{3/2} \rangle_{c.o} \simeq -(0.69 \pm 0.06) , \quad \langle \mathcal{O}_7^{3/2} \rangle_{c.o} \simeq -(0.11 \pm 0.01) , \quad (41)$$

where the errors come mainly from the small changes of  $t_c$ -values. If instead, we use the second set of values of  $t_c$  in Eq. (28), we obtain by setting  $\mu=2 \text{ GeV}$ :

$$\alpha_s \langle \mathcal{O}_8^{3/2} \rangle_{c.o} \simeq -(0.6 \pm 0.3) , \quad \langle \mathcal{O}_7^{3/2} \rangle_{c.o} \simeq -(0.10 \pm 0.03) , \quad (42)$$

which is consistent with the one in Eq. (41), but with larger errors as expected. We have also checked that both  $\langle \mathcal{O}_8^{3/2} \rangle$  and  $\langle \mathcal{O}_7^{3/2} \rangle$  increase in absolute value when  $\mu$  increases where a

<sup>10</sup>In the chiral limit  $f_\pi$  would be about 84 MeV. However, it is not clear to us what value of  $f_\pi$  should be used here, so we shall leave it as a free parameter which the reader can fix at his convenience.

stronger change is obtained for  $\langle \mathcal{O}_7^{3/2} \rangle$ , a feature which has been already noticed in [24]. In order to give a more conservative estimate, we consider as our final value the largest range spanned by our results from the two different sets of  $t_c$ -values. This corresponds to the one in Eq. (42) which is the less accurate prediction. We shall use the relation between the momentum cut-off (c.o) and  $\overline{MS}$ -schemes given in [25]:

$$\begin{aligned}\langle \mathcal{O}_7^{3/2} \rangle_{\overline{MS}} &\simeq \langle \mathcal{O}_7^{3/2} \rangle_{c.o} + \frac{3}{8}a_s \left( \frac{3}{2} + 2d_s \right) \langle \mathcal{O}_8^{3/2} \rangle \\ \langle \mathcal{O}_8^{3/2} \rangle_{\overline{MS}} &\simeq \left( 1 - \frac{119}{24}a_s \pm \left( \frac{119}{24}a_s \right)^2 \right) \langle \mathcal{O}_8^{3/2} \rangle_{c.o} - a_s \langle \mathcal{O}_7^{3/2} \rangle ,\end{aligned}\quad (43)$$

where  $d_s = -5/6$  (resp  $1/6$ ) in the so-called Naïve Dimensional Regularization NDR (resp. t'Hooft-Veltmann HV) schemes<sup>11</sup>;  $a_s \equiv \alpha_s/\pi$ . One can notice that the  $a_s$  coefficient is large in the 2nd relation (50% correction) [40], and the situation is worse because of the relative minus sign between the two contributions. Therefore, we have added a rough estimate of the  $a_s^2$  corrections based on the naïve growth of the PT series, which here gives 50% corrections of the sum of the two first terms. For a consistency of the whole approach, we shall use the value of  $\alpha_s$  obtained from  $\tau$ -decay, which is [30,31]:

$$\alpha_s(M_\tau)|_{exp} = 0.341 \pm 0.05 \implies \alpha_s(2 \text{ GeV}) \simeq 0.321 \pm 0.05 . \quad (44)$$

Then, we deduce (in units of  $10^{-4} \text{ GeV}^6$ ) at 2 GeV:

$$\langle \mathcal{O}_7^{3/2} \rangle_{\overline{MS}} \simeq -(0.7 \pm 0.2) , \quad \langle \mathcal{O}_8^{3/2} \rangle_{\overline{MS}} \simeq -(9.1 \pm 6.4) , \quad (45)$$

where the large error in  $\langle \mathcal{O}_8^{3/2} \rangle$  comes from the estimate of the  $a_s^2$  corrections appearing in Eq. (43). In terms of the  $B$  factor and with the previous value of the light quark masses in Eq. (21), this result, at  $\mu = 2 \text{ GeV}$ , can be translated into:

$$\begin{aligned}B_7^{3/2} &\simeq (0.7 \pm 0.2) \left( \frac{m_s(2) [\text{MeV}]}{119} \right)^2 \left( \frac{92.4}{f_\pi [\text{MeV}]} \right)^4 , \\ B_8^{3/2} &\simeq (2.5 \pm 1.3) \left( \frac{m_s(2) [\text{MeV}]}{119} \right)^2 \left( \frac{92.4}{f_\pi [\text{MeV}]} \right)^4 .\end{aligned}\quad (46)$$

- Our results in Eqs. (45) compare quite well with the ones obtained by [25] in the  $\overline{MS}$ -scheme (in units of  $10^{-4} \text{ GeV}^6$ ) at 2 GeV:

$$\langle \mathcal{O}_8^{3/2} \rangle_{\overline{MS}} \simeq -(6.7 \pm 0.9) , \quad \langle \mathcal{O}_7^{3/2} \rangle_{\overline{MS}} \simeq -(0.70 \pm 0.10) , \quad (47)$$

using the same sum rules but presumably a slightly different method for the uses of the data and for the choice of the cut-off in the evaluation of the spectral integral.

- Our errors in the evaluation of the spectral integrals, leading to the values in Eqs. (41) and (42), are mainly due to the slight change of the cut-off value  $t_c$ <sup>12</sup>.
- The error due to the passage into the  $\overline{MS}$ -scheme is due mainly to the truncation of the QCD series, and is important (50%) for  $\langle \mathcal{O}_8^{3/2} \rangle$  and  $B_8^{3/2}$ , which is the main source of errors in our estimate.

<sup>11</sup>The two schemes differ by the treatment of the  $\gamma_5$  matrix.

<sup>12</sup>A slight deviation from such a value affects notably previous predictions as the  $t_c$ -stability of the results ( $t_c \approx 2 \text{ GeV}^2$ ) does not coincide with the one required by the 2nd Weinberg sum rules. At the stability point the predictions are about a factor 3 higher than the one obtained previously.

- As noticed earlier, in the analysis of the pion mass-difference, it looks more natural to do the subtraction at  $t_c$ . We also found that moving the value of  $\mu$  can affects the value of  $B_{7,8}^{3/2}$ .

For the above reasons, we expect that the results given in [25] for  $\langle \mathcal{O}_8^{3/2} \rangle$  though interesting are quite fragile, while the errors quoted there have been presumably underestimated. Therefore, we think that a reconsideration of these results using alternative methods are mandatory.

## 7. $\langle (\pi\pi)_{I=2} | \mathcal{Q}_8^{3/2} | K^0 \rangle$ FROM THE HADRONIC TAU TOTAL DECAY RATES

In the following, we shall not introduce any new sum rule, but, instead, we shall exploit known informations from the total  $\tau$ -decay rate and available results from it, which have not the previous drawbacks. The  $V - A$  total  $\tau$ -decay rate, for the  $I = 1$  hadronic component, can be deduced from [35] (hereafter referred as BNP), and reads <sup>13</sup>:

$$R_{\tau, V-A} = \frac{3}{2} |V_{ud}|^2 S_{EW} \sum_{D=2,4,\dots} \delta_{V-A}^{(D)} . \quad (48)$$

$|V_{ud}| = 0.9753 \pm 0.0006$  is the CKM-mixing angle, while  $S_{EW} = 1.0194$  is the electroweak corrections [41]. In the following, we shall use the BNP results for  $\mathcal{R}_{\tau, V/A}$  in order to deduce  $R_{\tau, V-A}$ :

- The chiral invariant  $D = 2$  term due to a short distance tachyonic gluon mass [42,43] cancels in the  $V - A$  combination. Therefore, the  $D = 2$  contributions come only from the quark mass terms:

$$M_\tau^2 \delta_{V-A}^{(2)} \simeq 8 \left[ 1 + \frac{25}{3} a_s(M_\tau) \right] m_u(M_\tau) m_d(M_\tau) , \quad (49)$$

as can be obtained from the first calculation [32], where  $a_s(M_\tau) \equiv \alpha_s/\pi(M_\tau)$  and  $m_u(M_\tau) \simeq (3.5 \pm 0.4)$  MeV,  $m_d(M_\tau) \simeq (6.3 \pm 0.8)$  MeV [16] are respectively the running coupling and quark masses evaluated at the scale  $M_\tau$ .

- The dimension-four condensate contribution reads:

$$M_\tau^4 \delta_{V-A}^{(4)} \simeq 32\pi^2 \left( 1 + \frac{9}{2} a_s^2 \right) m_\pi^2 f_\pi^2 + \mathcal{O}(m_{u,d}^4) , \quad (50)$$

where we have used the  $SU(2)$  relation  $\langle \bar{u}u \rangle = \langle \bar{d}d \rangle$  and the Gell-Mann-Oakes-Renner PCAC relation:

$$(m_u + m_d) \langle \bar{u}u + \bar{d}d \rangle = -2m_\pi^2 f_\pi^2 . \quad (51)$$

- By inspecting the structure of the combination of dimension-six condensates entering in  $R_{\tau, V/A}$  given by BNP [35], which are renormalization group invariants, and using a  $SU(2)$  isospin rotation which relates the charged and neutral (axial)-vector currents, the  $D = 6$  contribution reads:

$$M_\tau^6 \delta_{V-A}^{(6)} = -2 \times 48\pi^4 a_s \left\{ \left( 1 + \frac{235}{48} a_s \pm \left( \frac{235}{48} a_s \right)^2 - \frac{\lambda^2}{M_\tau^2} \right) \langle \mathcal{O}_8^{3/2} \rangle + a_s \langle \mathcal{O}_7^{3/2} \rangle \right\} , \quad (52)$$

<sup>13</sup>Hereafter we shall work in the  $\overline{MS}$ -scheme.

where the overall factor 2 in front expresses the different normalization between the neutral isovector and charged currents used respectively in [25] and [35], whilst all quantities are evaluated at the scale  $\mu = M_\tau$ . The last two terms in the Wilson coefficients of  $\langle \mathcal{O}_8^{3/2} \rangle$  are new: the first term is an estimate of the NNLO term by assuming a naïve geometric growth of the  $a_s$  series; the second one is the effect of a tachyonic gluon mass introduced in [43], which takes into account the resummation of the QCD asymptotic series, with:  $a_s \lambda^2 \simeq -0.06 \text{ GeV}^2$ <sup>14</sup>. Using the values of  $\alpha_s(M_\tau)$  given previously, the corresponding QCD series behaves quite well as:

$$\text{Coefficient of } \langle \mathcal{O}_8^{3/2} \rangle \simeq 1 + (0.53 \pm 0.08) \pm 0.28 + 0.18 , \quad (53)$$

where the first error comes from the one of  $\alpha_s$ , while the second one is due to the unknown  $a_s^2$ -term, which introduces an uncertainty of 16% for the whole series. The last term is due to the tachyonic gluon mass. This leads to the numerical value:

$$M_\tau^6 \delta_{V-A}^{(6)} \simeq -(1.015 \pm 0.149) \times 10^3 \left\{ (1.71 \pm 0.29) \langle \mathcal{O}_8^{3/2} \rangle + a_s \langle \mathcal{O}_7^{3/2} \rangle \right\} , \quad (54)$$

- If, one estimates the  $D = 8$  contribution using a vacuum saturation assumption, the relevant  $V - A$  combination vanishes to leading order of the chiral symmetry breaking terms. Instead, we shall use the combined ALEPH/OPAL [30,31] fit for  $\delta_{V/A}^{(8)}$ , and deduce:

$$\delta_{V-A}^{(8)}|_{exp} = -(1.58 \pm 0.12) \times 10^{-2} . \quad (55)$$

We shall also use the combined ALEPH/OPAL data for  $R_{\tau,V/A}$ , in order to obtain:

$$R_{\tau,V-A}|_{exp} = (5.0 \pm 1.7) \times 10^{-2} , \quad (56)$$

Using the previous informations into the expression of the rate given in Eq. (48), one can deduce:

$$\delta_{V-A}^{(6)} \simeq (4.49 \pm 1.18) \times 10^{-2} . \quad (57)$$

This result is in good agreement with the result obtained by using the ALEPH/OPAL fitted mean value for  $\delta_{V/A}^{(6)}$ :

$$\delta_{V-A}^{(6)}|_{fit} \simeq (4.80 \pm 0.29) \times 10^{-2} . \quad (58)$$

We shall use as a final result the average of these two determinations, which coincides with the most precise one in Eq. (58). We shall also use the result:

$$\frac{\langle \mathcal{O}_7^{3/2} \rangle}{\langle \mathcal{O}_8^{3/2} \rangle} \simeq \frac{1}{8.3} \left( \text{resp. } \frac{3}{16} \right) , \quad (59)$$

where, for the first number we use the value of the ratio of  $B_7^{3/2}/B_8^{3/2}$  which is about  $0.7 \sim 0.8$  from e.g. lattice calculations quoted in Table 1, and the formulae in Eqs. (37) to (39); for the second number we use the vacuum saturation for the four-quark vacuum condensates [34]. The result in Eq. (59) is also comparable with the estimate of [25] from the sum rules given in Eq.(40). Therefore, at the scale  $\mu = M_\tau$ , Eqs. (52), (58) and (59) lead, in the  $\overline{MS}$ -scheme, to:

$$\langle \mathcal{O}_8^{3/2} \rangle (M_\tau) \simeq -(0.94 \pm 0.21) \times 10^{-3} \text{ GeV}^6 , \quad (60)$$

<sup>14</sup>This contribution may compete with the dimension-8 operators discussed in [44].

where the main errors come from the estimate of the unknown higher order radiative corrections. It is instructive to compare this result with the one using the vacuum saturation assumption for the four-quark condensate (see e.g. BNP):

$$\langle \mathcal{O}_8^{3/2} \rangle|_{v.s} \simeq -\frac{32}{18} \langle \bar{u}u \rangle^2 (M_\tau) \simeq -0.65 \times 10^{-3} \text{ GeV}^6, \quad (61)$$

which shows a  $1\sigma$  violation of this assumption. This result is not surprising as analogous violations have been obtained in other channels [36]. We have used for the estimate of  $\langle \bar{\psi}\psi \rangle$  the value of  $(m_u + m_d)(M_\tau) \simeq 10 \text{ MeV}$  [16] and the GMOR pion PCAC relation. However, this violation of the vacuum saturation is not quite surprising, as a similar fact has also been observed in other channels [36,30,31], though it also appears that the vacuum saturation gives a quite good approximate value of the ratio of the condensates [36,30,31]. The result in Eq. (60) is comparable with the value  $-(.98 \pm 0.26) \times 10^{-3} \text{ GeV}^6$  at  $\mu=2 \text{ GeV} \approx M_\tau$  obtained by [25] using a DMO-like sum rule, but, as discussed previously, the DMO-like sum rule result is very sensitive to the value of  $\mu$  if one fixes  $t_c$  as in Eq. (29) according to the criterion discussed above. Here, the choice  $\mu = M_\tau$  is well-defined, and then the result becomes more accurate (as mentioned previously our errors come mainly from the estimated unknown  $\alpha_s^3$  term of the QCD series). Using Eqs. (37) and (59), our previous result in Eq. (60) can be translated into the prediction on the weak matrix elements in the chiral limit:

$$\langle (\pi\pi)_{I=2} | \mathcal{Q}_8^{3/2} | K^0 \rangle (M_\tau^2) \simeq (2.58 \pm 0.58) \text{ GeV}^3 \left( \frac{92.4}{f_\pi [\text{MeV}]} \right)^3, \quad (62)$$

normalized to  $f_\pi$ , which avoids the ambiguity on the real value of  $f_\pi$  to be used in a such expression. Our result is higher by about a factor 2 than the quenched lattice result [17,9]. A resolution of this discrepancy can only be done after the inclusion of chiral corrections in Eqs. (37) to (39), and after the uses of dynamical fermions on the lattice. However, some parts of the chiral corrections in the estimate of the vacuum condensates are already included into the QCD expression of the  $\tau$ -decay rate and these corrections are negligibly small. We might expect that chiral corrections, which are smooth functions of  $m_\pi^2$  will not affect strongly the relation in Eqs. (37) to (39), though an evaluation of their exact size is mandatory. Using the previous mean values of the light quark running masses [16], we deduce in the chiral limit and at the scale  $M_\tau$ :

$$B_8^{3/2}(M_\tau^2) \simeq (1.70 \pm 0.39) \left( \frac{m_s(M_\tau) [\text{MeV}]}{119} \right)^2 \left( \frac{92.4}{f_\pi [\text{MeV}]} \right)^4. \quad (63)$$

One should notice that, contrary to the  $B$ -factor, the result in Eq. (62) is independent to leading order on value of the light quark masses.

## 8. NEW ALTERNATIVE SUM RULES FOR $\langle \mathcal{O}_7^{3/2} \rangle$

Here, we shall attempt to present new sum rules for extracting  $\mathcal{O}_7^{3/2}$ . In so doing, we work with the renormalized  $D=6$  condensate contributions to the difference of the vector and axial-vector two-point correlators. Using the expression given in the Appendix of BNP [35], one can deduce in the  $\overline{MS}$ -scheme and in the chiral limit:

$$\begin{aligned} (-q^2)^3 \Pi_{LR}(q^2) &= 4\pi\alpha_s(\mu) \left\{ \left( 1 + \frac{119}{24} a_s(\mu) \right) \langle \mathcal{O}_8^{3/2} \rangle + a_s(\mu) \langle \mathcal{O}_7^{3/2} \rangle \right\} \\ &\quad - 2\alpha_s^2(\mu) \left( \log \frac{-q^2}{\mu^2} \right) \left( -\mathcal{O}_8^{3/2} + \frac{8}{3} \mathcal{O}_7^{3/2} \right), \end{aligned} \quad (64)$$

which has the generic form:

$$q^6 \Pi_{LR}(q^2) \sim A \alpha_s(\mu) + B \alpha_s^2(\mu) + C \alpha_s^2(\mu) \log \frac{-q^2}{\mu^2}, \quad (65)$$

where, we remind that  $a_s \equiv \alpha_s/\pi$  is the renormalized coupling and  $A, B, C$  are constant numbers. One can notice that by working with the different  $q^2$ -derivatives of  $(-q^2)^3 \Pi_{V-A}(q^2)$ , one can eliminate the effects of the  $A$  and  $B$  terms, and derive the Laplace sum rule:

$$\tau \int_0^{t_c} ds s^3 e^{-s\tau} \frac{1}{\pi} \text{Im} \Pi_{LR}(s) \simeq -2\alpha_s^2(\tau)(1 - e^{-t_c\tau}) \left( -\langle \mathcal{O}_8^{3/2} \rangle + \frac{8}{3} \langle \mathcal{O}_7^{3/2} \rangle \right), \quad (66)$$

where we have transferred in the RHS the QCD continuum effect starting from the threshold  $t_c$ . Alternatively, we can also derive a  $\tau$ -like sum rule<sup>15</sup>:

$$\int_0^{t_c} \frac{ds}{t_c} s^3 \left(1 - \frac{s}{t_c}\right)^n \frac{1}{\pi} \text{Im} \Pi_{LR}(s) \simeq -2\alpha_s^2(t_c) \left(\frac{1}{n+1}\right) \left( -\langle \mathcal{O}_8^{3/2} \rangle + \frac{8}{3} \langle \mathcal{O}_7^{3/2} \rangle \right). \quad (67)$$

Formally, these sum rules are much better than the one proposed in [25,24], as the leading  $\mu$ -dependence has disappeared after taking different derivatives or after performing the Cauchy integral. However, unlike the ones in [25,24], one has gained one power of  $s$ , which renders the analysis more sensitive to the high-energy tail of the spectral functions where the data near the  $\tau$ -mass are quite bad [30,31]. One should also notice that here the RHS starts at order  $\alpha_s^2$ , which means that chiral corrections can affect dangerously the RHS of the sum rules, and can compete with the dimension-six condensate contributions. Using the present ALEPH/OPAL data on the  $V - A$  spectral functions, and a traditional  $\tau$ -stability analysis of the Laplace sum rule, we realize that the  $\tau$ -stability is reached at exceptional large  $\tau$ -values of about  $(1.8 \sim 3) \text{ GeV}^{-2} \approx M_\rho^{-2}$ , where the OPE can already break down. In the case of the  $\tau$ -like sum rule, one finds that for a given value of  $t_c$ , one has a  $n$ -stability where the value of  $n$  increases with  $t_c$  ( $n \geq 2.5$ ). However, at a such value of  $n$ , one needs more and more information (by duality) on the non-perturbative contributions to the sum rules. From our analysis and using the value of  $\langle \mathcal{O}_8^{3/2} \rangle$  obtained in Eq. (60), we obtain the conservative range:

$$-2 \times 10^{-3} \leq \langle \mathcal{O}_7^{3/2} \rangle \leq 10^{-2} \text{ GeV}^6, \quad (68)$$

where the lower (resp. higher) value corresponds to the first (resp. second) set of  $t_c$ -values given in Eq. (28). Therefore, we conclude that in order to extract more reliable informations on the previous sum rules, one needs to have much more informations on these sum rules both experimentally and theoretically. In particular, these sum rules can be more useful when accurate data near the  $\tau$ -mass is available and all chiral symmetry breaking terms are included both in the RHS of the sum rule and in the derivation of the relation between the kaon matrix elements and the vacuum condensates.

## 9. $I = 0$ SCALAR MESON CONTRIBUTION TO $\langle (\pi^+ \pi^-)_{I=0} | \mathcal{Q}_6^{1/2} | K_S^0 \rangle$

We study the effect of a direct production of a  $I = 0$  scalar meson intermediate state, which we shall denote by  $S_2 \equiv (\bar{u}u + \bar{d}d)$ , to the  $K_S^0 \rightarrow (\pi^+ \pi^-)_{I=0}$  decay process. Before doing our analysis, let us present the status of the scalar meson spectrum below 1 GeV.

### 9.1. A short review on the scalar meson spectrum below 1 GeV

A much more complete review and analysis of the complex structure of scalar mesons spectra is given in [46]. Here, we shall be concerned with the spectrum of light scalar mesons below 1 GeV:

$$a_0(980), \quad f_0(975), \quad \sigma(400 \sim 1200), \quad (69)$$

<sup>15</sup>The coefficient has been checked using a compilation of [45].

as quoted by PDG [4]. We associate the isovector state  $a_0(980)$  to the divergence of the vector current:

$$\partial_\mu V_{\bar{u}d}^\mu = (m_u - m_d)\bar{u}(i)d , \quad (70)$$

which is natural from the point of view of chiral symmetry [47,36,46,48] and in the construction of an effective chiral Lagrangian including resonances [49]. In this scheme the  $K_0^*(1430)$  [4] is the  $\bar{s}u$  partner of the  $a_0$ . The predicted hadronic and two-photon widths of the  $a_0$  using vertex sum rules [36,48,46] are in good agreement with present data [50,4]. These data indeed confirm the  $\bar{q}q$  nature of the  $a_0$  and disfavour some other exotic interpretations, which can further be tested through measurements of the  $\phi$  radiative decay at Daphne.

In our analysis of the  $I = 0$  isoscalar channel, we consider the trace of the energy momentum tensor:

$$\theta_\mu^\mu = \frac{1}{4}\beta(\alpha_s)G^2 + (1 + \gamma_m(\alpha_s)) \sum_{u,d,s} m_i \bar{\psi}_i \psi_i , \quad (71)$$

where  $\beta$  and  $\gamma_m$  are the  $\beta$  function and mass anomalous dimension. We shall consider the bare states before mixing:

- The meson  $S_2$  is associated to  $(\bar{u}u + \bar{d}d)$ , which is degenerate to the  $a_0$  because of the good  $SU(2)$  symmetry.
- The  $S_3 \equiv \bar{s}s$  state is above 1 GeV due to  $SU(3)$  breaking [36,46], and will not give significant effects in our analysis.
- The meson  $\sigma_B$  is a gluonium state, which has been needed for solving [51] the inconsistencies between the subtracted [53] and unsubtracted [54] gluonium sum rules<sup>16</sup>. The  $\sigma_B$  mass is expected to be around  $0.7 \sim 1$  GeV, but it will not play a significant rôle in this analysis, as the gluonium-quarkonium mixing in the propagator (mass mixing) via the off-diagonal two-point correlator is small [52,36].

The separation of the quark and gluon components of the current is allowed by renormalization group invariance (RGI) as  $m\bar{\psi}\psi$  is RGI, while  $\alpha_s G^2$  only mix with  $m\bar{\psi}\psi$  to higher order in  $\alpha_s$  [55]. The hadronic couplings, decay constants and masses of these mesons have been estimated using vertex sum rules [48,52,46], low-energy theorems [51,36,46], and/or some  $SU(3)$  symmetry relations among the meson wave functions [48]. It comes out that:

- The  $\sigma_B$  couples strongly and universally to pairs of Goldstone bosons (large violation of the OZI rule)<sup>17</sup> [51,36,46], which invalidates the lattice results in the quenched approximation<sup>18</sup>.
- The  $S_2$  is relatively narrow with a width of about 120 MeV and couples almost equally to  $\pi\pi$  and  $\bar{K}K$ .

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<sup>16</sup>The resolution of such inconsistencies has been also improved recently by the inclusion of the new  $1/q^2$ -term induced by the tachyonic gluon mass in the OPE [43].

<sup>17</sup>This “decay mixing” which occurs via 3-point function should not be confused with the “mass mixing” via an off-diagonal 2-point function. There is not a contradiction between a large “decay mixing” and a small “mass mixing”.

<sup>18</sup>The gluonium mass obtained in the quenched approximation of about 1.5 GeV can be identified with the one of about 1.6 GeV obtained from the unsubtracted sum rule [54,46], which is shown [51,46] to couple weakly to Goldstone pairs but strongly to glue rich  $U(1)_A$  states like  $\eta'-\eta'$ , through mixing to  $\eta'-\eta$ , and to  $4\pi$  through  $\sigma_B$ - $\sigma_B$  pairs.

- The  $I = 0$  scalar spectrum below 1 GeV, i.e., the observed wide  $\sigma$ -meson seen below 1 GeV [4,56] and the narrow  $f_0(980)$ [4] states, is expected in our approach to come from a maximal mixing between the  $S_2 \equiv (\bar{u}u + \bar{d}d)$  and the gluonium  $\sigma_B$  bare states.
- Recent data favour such a maximal gluonium-quarkonium mixing scheme [50], together with the  $\bar{q}q$  nature of the isovector  $a_0(980)$  state, though further refined tests are still needed.

## 9.2. Parameters of the $S_2 \equiv (\bar{u}u + \bar{d}d)$ scalar meson

In the following, we shall give the values of the decay constant and couplings of the  $S_2$  which is the relevant particle in the present analysis.

- Its decay constant has been fixed using Laplace sum rule for the associated two-point correlator. At the minimum of the sum rule variable  $\tau_0$ <sup>19</sup>, and at the inflexion point of its change versus the QCD continuum threshold of about 2.6 GeV<sup>2</sup>, it has the value [36,46]:

$$f_S/(m_u + m_d)(\tau_0) \simeq (0.32 \pm 0.08) , \quad (72)$$

where, presented in this way, the number in the RHS is not sensitive to the change of quark mass values, but has an anomalous dimension, and it runs like the inverse of the quark mass. It is normalized as:

$$\frac{1}{\sqrt{2}}(m_u + m_d)\langle 0|\bar{u}u + \bar{d}d|S_2\rangle = \sqrt{2}f_S M_S^2 . \quad (73)$$

One should notice that using the value of  $(m_u + m_d)(\tau_0)$  given in [16], the value of  $f_S$  is about 2 MeV, which is much smaller than  $f_\pi$ , and which invalidates the estimate  $f_S \approx f_\pi$  often proposed in the literature. Instead, it is the quantity  $M_S^2 f_S \approx m_\pi^2 f_\pi$ , which is almost constant.

- The  $S_2$  hadronic coupling to  $\pi\pi$  has been fixed using leading order results from vertex sum rules. It reads [48,52,36,46]:

$$g_{S\pi^+\pi^-} \approx \frac{16\pi^3}{3\sqrt{3}}\langle \bar{u}u \rangle \tau_0 \exp\left(M_S^2 \frac{\tau_0}{2}\right) \simeq 2.5 \text{ GeV}, \quad (74)$$

corresponding to  $\tau_0 \simeq 1 \text{ GeV}^{-2}$ . It leads to the decay width  $\Gamma(S_2 \rightarrow \pi^+\pi^-) \simeq 120 \text{ MeV}$ , with the normalization:

$$\Gamma(S_2 \rightarrow \pi^+\pi^-) = \frac{|g_{S\pi^+\pi^-}|^2}{16\pi M_S} \left(1 - \frac{4m_\pi^2}{M_S^2}\right)^{1/2} . \quad (75)$$

This result is in good agreement with the one obtained from the  $a_0\eta\pi$  coupling, by using  $SU(3)$  symmetry for the meson wave functions [48]:

$$g_{S\pi^+\pi^-} \simeq \sqrt{\frac{3}{2}}g_{a_0\eta\pi} \simeq (2.50 \pm 0.15) \text{ GeV}, \quad (76)$$

where we have used the peak data:  $\Gamma(a_0 \rightarrow \eta\pi) \simeq (57 \pm 7) \text{ MeV}$  [4]. One should notice that this coupling does not vanish in the chiral limit because it behaves like  $\langle \bar{u}u \rangle$ , and, up to  $SU(3)$ -breakings, one expects an universal coupling of the  $S_2$  to Goldstone boson pairs.

We shall use these parameters as inputs in the following analysis.

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<sup>19</sup>The inclusion of the effect of tachyonic gluon increases the value of  $\tau$  from 0.5 GeV<sup>-2</sup> [46,36] to 1 GeV<sup>-2</sup> [43], improving the duality between the resonance and the QCD sides of the sum rules, but does not almost affect the result, like in the case of the pion sum rule.



### 9.3. The $S_2 \equiv (\bar{u}u + \bar{d}d)$ scalar meson contribution to $K_S \rightarrow (\pi^+\pi^-)_I = 0$ decay

- We shall work with on-shell kaon, such that the tadpole diagram will not contribute in our analysis [57]. We can write, in the chiral limit:  $m_u = m_d = 0$  and  $\langle \bar{s}s \rangle = \langle \bar{d}d \rangle$ <sup>20</sup>:

$$\begin{aligned} \langle \mathcal{Q}_6^{1/2} \rangle_{2\pi} \equiv \langle (\pi^+\pi^-)_{I=0} | \mathcal{Q}_6^{1/2} | K^0 \rangle &\simeq - \left[ 2 \langle \pi^+ | \bar{u} \gamma_5 d | 0 \rangle \langle \pi^- | \bar{s} u | K^0 \rangle + \right. \\ &\quad \left. \langle \pi^+\pi^- | \bar{d}d + \bar{u}u | 0 \rangle \langle 0 | \bar{s} \gamma_5 d | K^0 \rangle \right]. \end{aligned} \quad (77)$$

For convenience, we shall evaluate the matrix elements at the scale  $\mu = m_c$  because the Wilson coefficients are also given at this scale [7]. We shall use the value of  $m_c$  from combined sum rule analysis of the charmonium and  $D$ -meson [58] and the QCD spectral sum rule average of the light quark masses quoted in [16]. They read at the scale  $\mu = m_c$ :

$$\begin{aligned} m_c(m_c) &\simeq (1.20 \pm 0.05) \text{ GeV}, \quad m_s(m_c) \simeq (147 \pm 15) \text{ MeV}, \\ m_d(m_c) &\simeq (8 \pm 1) \text{ MeV} \quad m_u(m_c) \simeq (4.4 \pm 0.5) \text{ MeV}. \end{aligned} \quad (78)$$

- The first term of the weak matrix element is well-known, and can be related to the  $K \rightarrow \pi \ell \nu_\ell$  semi-leptonic form factors (see e.g. [13]):

$$\langle \pi^- | \bar{s} u | K^0 \rangle = \left[ f_+(M_K^2 - m_\pi^2) + f_- m_\pi^2 \right] / (m_s - m_u), \quad (79)$$

with, to leading order in the chiral symmetry breaking terms:  $f_+ \approx 1$  and  $f_- \approx 0$ . It leads to:

$$\begin{aligned} \langle \pi^+ | \bar{u} \gamma_5 d | 0 \rangle \langle \pi^- | \bar{s} u | K^0 \rangle (m_c) &\simeq \frac{\sqrt{2} f_\pi m_\pi^2}{(m_d + m_u)} \frac{m_K^2 - m_\pi^2}{(m_s - m_u)} \\ &\simeq (0.323 \pm 0.032) \left( \frac{142.6}{(m_s - m_u) [\text{MeV}]} \right)^2 \text{ GeV}^3, \end{aligned} \quad (80)$$

where  $m_i$  are the running quark masses evaluated at the scale  $m_c$ . Chiral corrections to these terms are known to be about 10% in the literature (see e.g. [23]), which have been included into the error estimate.

- For the second term, we assume that it is dominated by the direct production of the  $S_2$ -scalar meson in the  $s$ -channel for an on-shell kaon  $p^2 = m_K^2$ . Therefore, it can be decomposed as:

$$\langle \pi^+\pi^- | \bar{d}d + \bar{u}u | 0 \rangle = \langle \pi^+\pi^- | S_2 \rangle \langle S_2 | \bar{d}d + \bar{u}u | 0 \rangle \equiv \frac{g_{S\pi^+\pi^-}}{(p^2 - M_{S_2}^2)} \frac{2f_S}{(m_u + m_d)} M_{S_2}^2. \quad (81)$$

With the values of the parameters given previously, we conclude that the scalar meson contribution is:

$$\langle \pi^+\pi^- | \bar{d}d + \bar{u}u | 0 \rangle \langle 0 | \bar{s} \gamma_5 d | K^0 \rangle (m_c) \simeq \pm (0.53 \pm 0.13) \left( \frac{155}{(m_s + m_d) [\text{MeV}]} \right) \text{ GeV}^3, \quad (82)$$

where we take into account the fact that QSSR cannot fix the signs of the  $S_2$  coupling and decay constant, which will be fixed later on from chiral constraints on the weak amplitude [59]. The error in this determination comes mainly from the one of decay constant  $f_S$ .

<sup>20</sup>We follow the notations and conventions of [13].

- For on-shell kaon, and neglecting, to a first approximation,  $m_{u,d}$  (resp.  $m_\pi^2$ ) versus  $m_s$  (resp.  $m_K^2$ ), we deduce from Eqs. (77) to (81), the approximate relation:

$$\langle \mathcal{Q}_6^{1/2} \rangle_{2\pi}(m_c) \approx -\frac{2\sqrt{2}f_\pi m_\pi^2}{(m_d + m_u)} \frac{m_K^2}{m_s} \left[ -1 + \frac{f_K}{f_\pi} \frac{g_{S\pi^+\pi^-} f_S}{m_\pi^2} \left( 1 - \frac{M_K^2}{M_S^2} \right)^{-1} \right], \quad (83)$$

which we can consider as an updated version of the expression given by VSZ in [13]. satisfying the double chiral constraint conditions (vanishing of the amplitude when  $M_K^2 \rightarrow 0$  and  $f_K = f_\pi$ ) [57], which are recovered if

$$\frac{g_{S\pi^+\pi^-} f_S}{m_\pi^2} \left( 1 - \frac{M_K^2}{M_S^2} \right)^{-1} \approx 1, \quad (84)$$

This value is obtained within the errors from the values of the  $S_2$ -parameters given previously.

- For the numerics, we shall use more precise values of the different parameters by keeping corrections to order  $m_\pi^2$  and  $m_{u,d}$ . Therefore, taking Eqs. (80) and (82) into Eq. (77), one obtains the final value of the weak matrix element:

$$\begin{aligned} \langle \mathcal{Q}_6^{1/2} \rangle_{2\pi}(m_c) \simeq & - \left( \frac{142.6}{(m_s - m_u) [\text{MeV}]} \right)^2 \times \left[ (0.65 \pm 0.09) \right. \\ & \left. - (0.53 \pm 0.13) \left( \frac{(m_s - m_u) [\text{MeV}]}{142.6} \right) \right] \text{GeV}^3. \end{aligned} \quad (85)$$

Using the relation [7]<sup>21</sup>:

$$\langle \mathcal{Q}_6^{1/2} \rangle_{2\pi}(m_c) \simeq -4\sqrt{\frac{3}{2}} \left( \frac{m_K^2}{m_s + m_d} \right)^2 \sqrt{2} (f_K - f_\pi) B_6^{1/2}(m_c), \quad (86)$$

where  $f_K \simeq 1.22f_\pi$ , the previous result can be translated into:

$$B_6^{1/2}(m_c) \simeq 3.7 \left( \frac{m_s + m_d}{m_s - m_u} \right)^2 \left[ (0.65 \pm 0.09) - (0.53 \pm 0.13) \left( \frac{(m_s - m_u) [\text{MeV}]}{142.6} \right) \right] \quad (87)$$

Evaluating the running quark masses at 2 GeV, with the values given previously, one deduces:

$$\begin{aligned} B_6^{1/2}(2) & \simeq (1.0 \pm 0.4) \text{ for } m_s(2) = 119 \text{ MeV}, \\ & \leq (1.5 \pm 0.4) \text{ for } m_s(2) \geq 90 \text{ MeV}. \end{aligned} \quad (88)$$

The errors added quadratically have been relatively enhanced by the partial cancellations of the two contributions.

## 10. COMPARISON OF OUR RESULTS WITH SOME OTHER PREDICTIONS

In this section, we shall compare our values of  $B_{7,8}^{3/2}$  and  $B_6^{1/2}$  with the results in Table 1.

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<sup>21</sup>We shall use the usual parametrization in terms of  $m_s^2$ , but it can be misleading in view of the  $m_s$ -dependence of our results in Eqs. (80) and (82).

### 10.1. Value of $B_7^{3/2}$

- Our value in Eq. (46):

$$B_7^{3/2}(\mu = 2 \text{ GeV}) \simeq (0.7 \pm 0.2) \left( \frac{m_s(2) [\text{MeV}]}{119} \right)^2 \left( \frac{92.4}{f_\pi [\text{MeV}]} \right)^4, \quad (89)$$

comes from a re-analysis of the DMO-like sum rule used in [25], and in [24] within a large  $N_c$  expansion and a lowest meson dominance (LMD). One can notice in Table 1 a quite good agreement between the results from different approaches. However, due to the strong  $\mu$ -dependence of the result, one should be careful when giving its value.

- Our analysis from the Laplace and  $\tau$ -like sum rules are unfortunately inconclusive using present  $\tau$ -decay data and present theoretical approximation (chiral limit).

### 10.2. Value of $B_8^{3/2}$

Our result in Eq. (63):

$$B_8^{3/2}(M_\tau^2) \simeq (1.7 \pm 0.4) \left( \frac{m_s(M_\tau) [\text{MeV}]}{119} \right)^2 \left( \frac{92.4}{f_\pi [\text{MeV}]} \right)^4, \quad (90)$$

comes from the analysis of the total  $\tau$  hadronic width  $R_{\tau,V-A}$ .

- The closest comparison to be made is the one with [25] where soft pion and kaon reductions together with DMO sum rules have been used. We have stressed that in order to obtain our value, we have not introduced any new sum rule but took advantage of the existing measurement of the vector and axial-vector components of the  $\tau$ -total width and the measured values of the corresponding  $D = 6$  vacuum condensates.
- Our result agrees numerically within the errors with the one of [25], but we have also shown that the DMO sum rules lead to an inaccurate value due mainly to the bad convergence of the QCD series.
- Our result includes NLO corrections, an estimate of the NNLO terms and the effect of a tachyonic gluon mass which phenomenologically takes into account the resummation of the QCD asymptotic series.
- Our result is in the range given by some linear [22] and non-linear [23]  $\sigma$  models, but is 1 to  $2\sigma$  higher than the largest values obtained from the present lattice [9,18], large  $N_c$  [20], chiral quark model [10] and the one including final state interactions [26]. Though the agreements with the results from the  $\sigma$  models are interesting, it is not clear to us how to connect the two approaches, as in these models, an  $I = 0$  scalar resonance has been introduced with the parameters of the observed  $\sigma$ -meson which we expect [51,36,46] to have a large gluon component in its wave function. The difference with the result from final state interactions [26] is more rewarding, and needs a much better understanding in connection to the comments raised in [27].

### 10.3. Value of $B_6^{1/2}$

Assuming a dominance of the  $I = 0$   $S_2 \equiv (\bar{u}u + \bar{d}d)$  scalar meson contribution, through the operator  $\mathcal{Q}_6^{1/2}$ , to the  $K_S \rightarrow (\pi^+\pi^-)_I = 0$  decay amplitude, we have obtained:

$$B_6^{1/2}(m_c) \simeq 3.7 \left( \frac{m_s + m_d}{m_s - m_u} \right)^2 \left[ (0.65 \pm 0.09) - (0.53 \pm 0.13) \left( \frac{(m_s - m_u) [\text{MeV}]}{142.6} \right) \right], \quad (91)$$

leading to:

$$\begin{aligned} B_6^{1/2}(2) &\simeq (1.0 \pm 0.4) \quad \text{for } m_s(2) = 119 \text{ MeV}, \\ &\leq (1.5 \pm 0.4) \quad \text{for } m_s(2) \geq 90 \text{ MeV}, \end{aligned} \tag{92}$$

which we give in Table 1.

- Contrary to *conventional* approaches [1,7,9,13], which do not consider the effect of a scalar meson, our result shows that the  $\bar{q}q$  component of the scalar meson tends to bring the value of  $B_6^{1/2}$  to the one of the leading  $1/N_c$  expectation.
- In our analysis, the mass of the  $S_2$  is fixed from  $SU(2)$  symmetry arguments, and obtained from the two-point function sum rule to be about the one of the  $a_0(980)$ , which is relatively high compared with the kaon mass. Therefore, the main contribution observed here comes from the values of the  $S_2$ -coupling to  $\pi\pi$  and of its decay constant. However, for a more definite conclusion, it is important to look for the effect of the gluon component of the  $\sigma$ , which can eventually give a sizeable contribution in the amplitude through a new operator other than the one discussed here. This new feature may clarify the observed enhancement from a direct final state interactions (FSI) analysis of the amplitude. We plan to come back to this point in a future work.
- Present lattice results [9,18] are still unreliable [19], as the NLO QCD corrections at the matching scale between the lattice and continuum results are huge. Measuring the effects of the scalar meson on the lattice seems to be difficult due to the propagator [60].
- Due to the partial cancellations of the two contributions in the weak amplitude, taking into account the alone effect of the  $\bar{q}q$  component of the isoscalar meson is not sufficient for explaining the large enhancement obtained from some other approaches quoted in Table 1 which we list below:
  - Some incomplete large  $N_c$  result [20] including higher order corrections  $\mathcal{O}(p^2/N_c)$  in the chiral limit.
  - A version of the ENJL-model [1] with an intermediate vector bosons [21] and the chiral quark model [10] where both models are based on the  $1/N_c$  expansion. However, for the chiral quark model the predictions correspond to a lower value of the scale  $\mu$ . A clear connection with these results with the  $\overline{MS}$  scheme as well as the relation of the parameters used there with lattice and QSSR calculations is needed.
  - Final state interactions [26] where there is still a debate for fixing the slope of the amplitude [27].
  - Enhancements due to the isoscalar meson have also been found from the  $\sigma$ -model approaches [23,22], which is mainly due to the small value of the  $\sigma$ -mass used in the  $\sigma$ -propagator appearing in Eq. (81). However, the uncertainties come from the fact that, in the linear sigma models [22] the Lagrangian is not unique, while in the non-linear  $\sigma$  models [23], the  $\sigma$  mass is a free parameter which is usually identified with the observed wide  $\sigma$ -meson having a mass in the range  $(0.4 \sim 1.2)$  GeV and a width of about  $(600 \sim 1000)$  MeV [4,56]. On the contrary, in the present work, for the reasons previously explained, the scalar meson entering into the analysis is not the observed  $\sigma$  where, within our framework, the  $\sigma$  comes from a maximal (decay) mixing between a  $S_2$  ( $\bar{q}q$ ) and gluonium ( $\sigma_B$ ) states. Indeed, as explained in the previous sections, only the  $\bar{q}q$  component (the hypothetical  $S_2$  state) of the  $\sigma$  is relevant for the present

operator <sup>22</sup>. However, some advanced versions of the effective Lagrangian approach, which can separate explicitly the  $\bar{q}q$  from the gluon component of the scalar meson, are needed for the present problem <sup>23</sup>.

- A clever explanation showing the connections of the different results reviewed here in Table 1 in order to have an unified explanation of these different determinations is still needed.

#### 10.4. VALUES OF $B_6^{1/2}/B_8^{3/2}$

Using our previous determinations of  $B_8^{3/2}(M_\tau)$  in Eq. (63) and  $B_6^{1/2}(2)$  in Eq. (88), and the previous value of  $m_s(2)$ , one can deduce the ratio:

$$\mathcal{R}_{68} \equiv \frac{B_6^{1/2}}{B_8^{3/2}} \simeq 0.6 \pm 0.3 , \quad (93)$$

and their combination:

$$\mathcal{B}_{68} \equiv B_6^{3/2} - 0.48B_8^{3/2} \simeq (0.3 \pm 0.4) \text{ for } m_s(2) = 119 \text{ MeV} , \quad (94)$$

where we have added the errors quadratically.

Instead, using the lower bound  $m_s(2) \geq 90 \text{ MeV}$  reported in [16] into the expressions of  $B_8^{3/2}(M_\tau)$  in Eq. (63) and  $B_6^{1/2}(2)$ , one can deduce the *conservative* upper bound:

$$\mathcal{B}_{68} \equiv B_6^{3/2} - 0.48B_8^{3/2} \leq (1.0 \pm 0.4) \text{ for } m_s(2) \geq 90 \text{ MeV} , \quad (95)$$

where again we have added the errors quadratically.

#### 11. VALUE AND UPPER BOUND OF $\epsilon'/\epsilon$

- The estimated value in Eq. (94) does not satisfy the constraint required in Eq. (22) for explaining the present data on the  $CP$ -violation ratio  $\epsilon'/\epsilon$  given in Eq. (10). It leads, for  $m_s(2) = 119 \text{ MeV}$ , to the prediction:

$$\frac{\epsilon'}{\epsilon} \simeq (4 \pm 5) \times 10^{-4} . \quad (96)$$

The failure for reproducing the data may indicate the need for other contributions than the alone  $\bar{q}q$  scalar meson  $S_2$  (not the observed  $\sigma$ )-meson for explaining, within the standard model (SM), these data. Among others, a much better understanding of the effects of the gluonium (expected large component of the  $\sigma$ -meson [51,46,48]) in the amplitude, through presumably a new operator needs to be considered.

- If we use instead the *conservative* upper bound for  $\mathcal{B}_{68}$  in Eq. (95) corresponding to the lower bound for the strange quark mass  $m_s(2) \geq 90 \text{ MeV}$ , we can deduce the bound <sup>24</sup>:

$$\frac{\epsilon'}{\epsilon} \leq (22 \pm 9) \times 10^{-4} . \quad (97)$$

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<sup>22</sup>The effect of the gluon component (called  $\sigma_B$ ) through the scalar propagator is negligible (mass mixing) [52,36,46] as it comes from the off-diagonal quark-gluon two-point function. This fact does not contradict the large gluonium decay into  $\pi\pi$  (decay mixing) which comes from a vertex function [51,36,46].

<sup>23</sup>Some attempts to introduce the  $I = 0$  scalar meson within the effective lagrangian framework exist in the literature [49,23,22,61].

<sup>24</sup>The real value of  $f_\pi$  (92.4 or 87 MeV) used in the chiral limit expression of  $B_8^{3/2}$  does not affect significantly the result.

The errors come mainly from  $\mathcal{B}_{68}$  (40%) and  $\text{Im } \lambda_t$  (10.5%), which we have added quadratically. In  $B_6^{1/2}$ , the large error is due to the partial cancellation of the contributions from the semi-leptonic form factors and the  $S_2$  resonance. This bound agrees within the errors with the data on the  $CP$ -violation ratio  $\epsilon'/\epsilon$  given in Eq. (10).

## 12. COMMENTS ON THE $\Delta I = 1/2$ RULE

However, unlike Ref. [22], we do not expect that our result will affect significantly the  $CP$ -conserving  $\Delta I = 1/2$  rule process. Indeed, according to the analysis in [1,7], the amplitude  $\text{Re}A_0$  of this process is dominated by the pure  $I = 1/2$  combination:

$$Q_- \equiv Q_2 - Q_1, \quad (98)$$

where its Wilson coefficient is relatively enhanced compared with the ones of  $Q_+ \equiv Q_2 + Q_1$  and  $Q_6$ . Moreover, the one of  $Q_6$ , where the  $S_2 \equiv (\bar{u}u + \bar{d}d)$  can contribute, can even be zero at the subtraction point  $\mu = m_c$  for a given renormalization scheme (so-called HV-scheme), but still remains negligible at larger values of  $\mu$  where the perturbative calculations of these Wilson coefficients can be trusted. Instead, octet scalar may play a role in this process as has been emphasized in the first study of this process on the lattice [62]. We plan to analyze carefully this process in a future work.

## 13. SUMMARY

- We have used the recent ALEPH/OPAL data on the  $V - A$  hadronic spectral functions from  $\tau$ -decays for fixing the matching scale separating the low and high-energy regions (continuum threshold), at which the first and second Weinberg sum rules should be realized in the chiral limit [Eq. (29)]. We have used this information for predicting and for testing the accuracy of the low-energy constants  $m_{\pi^+} - m_{\pi^0}$  [Eq. (31)] and  $L_{10}$  [Eq. (30)], and the electroweak kaon penguin matrix elements  $\langle(\pi\pi)_{I=2}|\mathcal{Q}_{8,7}^{3/2}|K^0\rangle$  [Eqs. (45) and (46)] obtained from DMO-like sum rules in the chiral limit.
- We have estimated the value of the weak matrix element  $\langle(\pi\pi)_{I=2}|\mathcal{Q}_8^{3/2}|K^0\rangle$  using the measured  $V/A$   $\tau$ -decay rate and the experimentally fitted value of the dimension six-operators, without introducing any additional sum rules. Our results, in Eqs. (62) and (63), indicate a deviation from the vacuum saturation of the four-quark condensates, where analogous violations have been already found from the analysis of other channels [30,31,36].
- We have introduced some alternative new sum rules in Eqs (66) and (67) in order to estimate  $\langle(\pi\pi)_{I=2}|\mathcal{Q}_7^{3/2}|K^0\rangle$ , which require improved data in the region near the  $\tau$ -lepton mass and more theoretical inputs in order to be useful. At present, the result from the DMO-like sum rule obtained in Eq. (46) is more meaningful.
- We remind that the results for  $\langle(\pi\pi)_{I=2}|\mathcal{Q}_{8,7}^{3/2}|K^0\rangle$  matrix elements have been obtained in the chiral limit as we have taken advantage of the soft pion and kaon reductions techniques in order to express them in terms of the vacuum condensates. Main improvements of our results need the inclusion of these chiral corrections.
- In the last part of the paper, we have analyzed the effect of the  $S_2 \equiv (\bar{u}u + \bar{d}d)$  component of the  $I = 0$  scalar meson into the  $\langle(\pi\pi)_{I=0}|\mathcal{Q}_6^{1/2}|K^0\rangle$  matrix element. We found that its main contribution is due to the values of its decay constant and coupling to  $\pi\pi$  giving the predictions in Eqs. (85) and (91), but not on the enhancement due to its mass in the propagator.

- We have expressed our predictions on the matrix elements in terms of the  $B$ -parameters, with the absolute values given in Eqs. (63), (91) and rescaled at 2 GeV in Table 1, where we have used the value  $m_s(2 \text{ GeV}) \simeq (119 \pm 12) \text{ MeV}$  [16]. These results are compared with the ones from different approaches. The ratio of these  $B$ -parameters is also given in Eq. (93) and compared with the existing values. Their combination given in Eq. (94) is also compared with the constraint in Eq. (22).
- We have used our previous determinations of the penguin matrix elements in order to predict the value of the  $CP$ -violating ratio  $\epsilon'/\epsilon$ . Our conservative upper bound in Eq. (97) corresponding to  $m_s(2) \geq 90 \text{ MeV}$ , agrees with the present day experiments given in Eq. (10). However, our estimate in Eq. (96) corresponding to  $m_s(2) \simeq 119 \text{ MeV}$  fails to explain the data, which is mainly due to the partial cancellation of the contributions implied by the chiral constraints governing the  $\langle (\pi\pi)_{I=0} | \mathcal{Q}_6^{1/2} | K^0 \rangle$  matrix element. This failure may not be quite surprising as the observed  $\sigma$ -meson is expected to have a large gluonium admixture responsible for its large  $\pi\pi$  width [51,46,48], which can manifest in the  $K \rightarrow \pi\pi$  amplitude through an eventual new operator not considered until now, but most probably along the line of dimension-8 operators discussed recently [44]. Further study of the gluonium effect is therefore mandatory before an eventual consideration of possible effects due to new physics.

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